

SYMPLECTIC BIRATIONAL GEOMETRY

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Dedicated to the occasion of Yasha Eliashberg's 60-th birthday

1. INTRODUCTION

Birational geometry has always been a fundamental topic in algebraic geometry. In the 80's, an industry called Mori's birational geometry program was created for the birational classification of algebraic manifolds of dimension three. Roughly speaking, the idea of Mori's program is to divide algebraic varieties into two categories: uniruled versus non-uniruled. The uniruled varieties are those containing a rational curve through every point. Even for this class of algebraic manifolds, the classification is usually not easy. So one is content to carry out the classification of the more restrictive Fano manifolds and prove some structural theorems such as the Mori fiber space structure of any uniruled variety with Fano fibers. For non-uniruled manifolds, one wishes to construct a "minimal model" by a sequence of contraction analogous to the blow-downs. One immediate problem is that a contraction of smooth variety often results a singular variety. This technical problem often makes the subject of birational geometry quite difficult. Only recently, the minimal model program was carried out to a large extent in higher dimensions in the remarkable papers [1] and [56].

In the early 90's, the second author observed that some aspects of this extremely rich program of Mori can be extended to symplectic geometry via the newly created Gromov-Witten theory [50]. Specifically, he extended Mori's notion of extremal rays to the symplectic category and used it to study the symplectomorphism group. A few years later, Kollár and the second author showed that a smooth projective uniruled manifold carries a non-zero genus zero Gromov-Witten invariant with a point insertion. Shortly after, further deep relations between the Gromov-Witten theory and birational geometry were discovered in [51], resulting in the speculation that there should be a symplectic birational geometry program. In the meantime the Gromov-Witten theory, together with the Seiberg-Witten theory, was applied with spectacular success to obtain basic structure theorems of symplectic 4-manifolds, especially the rational and ruled ones, cf. [43], [57], [36], [29].

To be more specific, we define a genus 0 GW class as a nonzero degree 2 homology class supporting nontrivial genus 0 GW invariants. Let us then

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ask the following question: *what kind of structures of a symplectic manifold are detected by its genus zero GW classes?* What we would like to convey in this article is that the answer is precisely the *symplectic birational structure!* To start with consider the sweep out of all the pseudo-holomorphic rational curves in a GW class. The extreme case is that the sweep out is the whole space. In this case the manifold is likely a *uniruled manifold*. In general, the sweep out is likely a possibly singular *uniruled submanifold*. It has been gradually realized that the symplectic birational geometry deals precisely with these uniruled manifolds and uniruled submanifolds.

In this article, we outline the main elements of this new program in symplectic geometry. Let us first mention that some technical difficulties in the algebraic birational geometry are still present in our program, but we might be able to treat them with more symplectic topological techniques. This is certainly the case for contractions. Recall that the goal of birational geometry is to classify algebraic varieties in the same birational class. Two algebraic varieties are birational to each other if and only if there is a *birational map* between them. A birational map is an isomorphism between Zariski open sets, but it is not necessarily defined everywhere. If a birational map is defined everywhere, we call it a contraction. A contraction changes a lower dimensional uniruled subvariety only, hence we can view it as a surgery. Intuitively, a contraction simplifies a smooth variety, but as already mentioned, it often produces a singular variety. Various other types of surgeries are needed to deal with the resulting singularities. The famous ones are flip and flops which are much more subtle than contractions. We certainly can not avoid some aspects of this issue in our program. A major problem in our program is then to see whether the flexibility in the symplectic category can produce many such kinds of surgery operations. In particular, we would like to interpret and construct flips and flops symplectically.

A new phenomenon in our program is that many obvious properties of algebraic birational geometry are no longer obvious in the symplectic category. Notably, the birational invariance of uniruledness in [13] is such an example, where we have to draw newly developed powerful technology from the Gromov-Witten theory.

However, this perspective makes the subject distinctively symplectic. And despite of these ‘old’ and ‘new’ obstacles, major progress has been made recently in [13], [31], [32], [45].

One eventual and remote goal of symplectic geometry is to classify symplectic manifolds. Symplectic birational geometry can be considered as the first step towards such a classification [51]. In addition, the author hope that symplectic topological techniques and view points in this new program will also bring some fresh insight to the birational classification of algebraic manifolds.

The article is organized as follows. We will set up symplectic birational equivalence in section two. The transition as an extended symplectic birational transformation will also be discussed. Section three is devoted to the

birational invariance of uniruled manifolds and its classifications. In section four, we will discuss the dichotomy of uniruled submanifolds. In section five, we will briefly discuss speculations on minimal symplectic manifolds. We describe various GW correspondences in section six. We finish the paper by several concluding remarks. We should mention that this article does not contain any proofs but provide appropriate references for the results discussed.

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2. BIRATIONAL EQUIVALENCE IN SYMPLECTIC GEOMETRY

For a long time, it was not really clear what is an appropriate notion of birational equivalence in symplectic geometry. Simple birational operations such as blow-up/blow-down were known in symplectic geometry for a long time [12, 47]. But there is no straightforward generalization of the notion of a general birational map in the flexible symplectic category. The situation changed a great deal when the weak factorization theorem was established recently (see the lecture notes [39] and the reference therein) that any birational map between projective manifolds can be decomposed as a sequence of blow-ups and blow-downs. This fundamental result resonates perfectly with the picture of the wall crossing of symplectic reductions analyzed by Guillemin-Sternberg in the 80's. Therefore, we propose to use their notion of cobordism in [12] as the symplectic analogue of the birational equivalence (see Definition 2.1). To avoid confusion with other notions of cobordism in the symplectic category, we would call it *symplectic birational cobordism*.

2.1. Birational equivalence. The basic reference for this section is [12]. We start with the definition which is essentially contained in [12].

Definition 2.1. *Two symplectic manifolds (X, ω) and (X', ω') are birational cobordant if there are a finite number of symplectic manifolds (X_i, ω_i) , $0 \leq i \leq k$, with $(X_0, \omega_0) = (X, \omega)$ and $(X_k, \omega_k) = (X', \omega')$, and for each i , (X_i, ω_i) and (X_{i+1}, ω_{i+1}) are symplectic reductions of a semi-free Hamiltonian S^1 symplectic manifold W_i (of 2 dimensions higher).*

Here an S^1 action is called semi-free if it is free away from the fixed point set.

There is a related notion in dimension 4 in [48]. However we remark that the cobordism relation studied in this paper is quite different from some other notions of symplectic cobordisms, see [4], [5], [8], [9].

According to [12], we have the following basic factorization result.

Theorem 2.2. *A birational cobordism can be decomposed as a sequence of elementary ones, which are modeled on blow-up, blow-down and \mathbb{Z} -linear deformation of symplectic structure.*

A \mathbb{Z} -linear deformation is a path of symplectic form $\omega + t\kappa$, $t \in I$, where κ is a closed 2-form representing an integral class and I is an interval. It was shown in [13] that \mathbb{Z} -linear deformations are essentially the same as general deformations.

Observe that a polarization on a projective manifold, which is simply a very ample line bundle, gives rise to a symplectic form with integral class, well-defined up to isotopy. Together with the weak factorization theorem mentioned in the previous page, we then have

Theorem 2.3. *Two birational projective manifolds with any polarizations are birational cobordant as symplectic manifolds.*

3. UNIRULED SYMPLECTIC MANIFOLD

3.1. Basic definitions and properties. Let us first recall the notion of uniruledness in algebraic geometry.

Definition 3.1. *A projective manifold X (over \mathbb{C}) is called (projectively) uniruled if for every $x \in X$ there is a morphism $f : \mathbb{P}^1 \rightarrow X$ satisfying $x \in f(\mathbb{P}^1)$, i.e. X is covered by rational curves.*

A beautiful property of a uniruled projective manifold ([18], [17]) is that general rational curves are unobstructed, hence regular in the sense of the Gromov-Witten theory. It is this property which underlies the aforementioned result of Kollár and Ruan (stated here in a sharper form noticed by McDuff, cf. [31]).

Theorem 3.2. *A projective manifold is projectively uniruled if and only if*

$$\langle [pt], [\omega]^p, [\omega]^q \rangle_A^X > 0$$

for a nonzero class A , a Kähler form ω and integers p, q . Here $[pt]$ denotes the fundamental cohomology class of X .

In this article, for a closed symplectic manifold (X, ω) , we denote its genus zero GW invariant in the curve class $A \in H_2(X; \mathbb{Z})$ with cohomology constraints $\alpha_1, \dots, \alpha_k \in H^*(X; \mathbb{R})$ by

$$(1) \quad \langle \alpha_1, \dots, \alpha_k \rangle_A^X.$$

To define it we need to first choose an ω -tamed almost complex structure J and consider the moduli space of J -holomorphic rational curves with k marked points in the class A . Via the evaluation maps at the marked points we pull back α_i to cohomology classes over the moduli space, and the invariant (1) is supposed to be the integral of the cup product of the pull-back classes over the moduli space. Due to compactness and transversality issues the actual definition requires a great deal of work ([6], [38], [54], [53]). Intuitively the GW invariant (1) counts J -holomorphic rational curves in the class A passing through cycles Poincaré dual to α_i .

Definition 3.3. *Let $A \in H_2(X; \mathbb{Z})$ be a nonzero class. A is called a GW class if there is a non-trivial genus zero GW invariant of (X, ω) with curve class A . A is said to be a uniruled class if it is a GW class and moreover, there is a nonzero GW invariant of the form*

$$(2) \quad \langle [pt], \alpha_2, \dots, \alpha_k \rangle_A^X,$$

where $\alpha_i \in H^*(X; \mathbb{R})$. X is said to be (symplectically) uniruled if there is a uniruled class.

Remark 3.4. *It is easy to see that we could well use the GW invariants with a disconnected domain to define this concept, subject to the requirement that the curve component with the $[pt]$ constraint represent nonzero class in $H_2(X; \mathbb{Z})$. This flexibility is important for the proof of the birational cobordism invariance.*

This notion has been studied in the symplectic context by G. Lu (see [34], [35]). Notice that, by [25], it is not meaningful to define this notion by requiring that there is a symplectic sphere in a fixed class through every point, otherwise every simply connected manifold would be uniruled.

Remark 3.5. *According to Theorem 3.2 a projectively uniruled manifold is symplectically uniruled, in fact strongly symplectically uniruled. Here X is said to be strongly uniruled if there is a nonzero invariant of the form (2) with $k = 3$.*

Obviously the only uniruled 2-manifold is S^2 . In dimension 4 the converse is essentially true (see Theorem 4.3). While in higher dimensions it follows from [10] (see also [35]) that there are uniruled symplectic manifolds which are not projective, and it follows from [52] that there could be infinitely many distinct uniruled symplectic structures on a given smooth manifold.

There are also descendant GW invariants which are variations of the GW invariants with constraints of the form $\tau_{j_i}(\alpha_i)$. Here the class $\tau_{j_i}(\alpha_i)$ over the moduli space is the cup product of the pull-back of the class α_i and the $j_i - th$ power of a natural degree 2 class, which is the 1st Chern class of the (orbifold) line bundle over the moduli space whose fibers are the cotangent lines at the $i - th$ marked point.

It is very useful to characterize uniruledness using these more general GW invariants ([13]).

Theorem 3.6. *A symplectic manifold X is uniruled if and only if there is a nonzero, possibly disconnected genus zero descendant GW invariant*

$$(3) \quad \langle \tau_{j_1}([pt]), \tau_{j_2}(\alpha_2), \dots, \tau_{j_k}(\alpha_k) \rangle_A^X$$

such that the curve component with the $[pt]$ constraint has nonzero curve class.

In particular, Theorem 3.6 is used to establish the fundamental birational invariance property of uniruled manifolds in [13].

Theorem 3.7. *Symplectic uniruledness is invariant under symplectic blow-up and blow-down.*

3.2. Constructions. An important aspect of symplectic birational geometry is the classification of uniruled manifolds. This remains to be a distant goal. A more immediate problem is to construct more examples. Kollár-Ruan's theorem shows that all the algebraic uniruled manifold is symplectic uniruled. Another class of example is from the following beautiful theorem of McDuff in [45].

Theorem 3.8. *Any Hamiltonian S^1 -manifold is uniruled. Here a Hamiltonian S^1 -manifold is a symplectic manifold admitting a Hamiltonian S^1 -action.*

A rich source of uniruled manifolds comes from almost complex uniruled fibrations. Suppose that $\pi : X \rightarrow B$ is a fibration (with possibly singular fibers) where X and B are symplectic manifolds. We call it *an almost complex fibration* if there are tamed J, J' for X, B such that π is almost complex. Symplectic fiber bundles over symplectic manifolds in the sense of Thurston are almost complex fibrations. Lefschetz fibrations, or more generally, locally holomorphic fibrations studied in [11] are also almost complex.

Let $\iota : \pi^{-1}(b) \rightarrow X$ be the embedding for a smooth fiber over $b \in B$. We have the following result in [31] by a direct geometric argument.

Proposition 3.9. *Suppose that $\pi : X \rightarrow B$ is an almost complex fibration between symplectic manifolds X, B . Then, for $A \in H_2(\pi^{-1}(b); \mathbb{Z})$ and $\alpha_2, \dots, \alpha_k \in H^*(X; \mathbb{R})$,*

$$(4) \quad \langle [pt], \iota^* \alpha_2, \dots, \iota^* \alpha_k \rangle_A^{\pi^{-1}(b)} = \langle [pt], \alpha_2, \dots, \alpha_k \rangle_{\iota_*(A)}^X.$$

Corollary 3.10. *Suppose that $\pi : X \rightarrow B$ is an almost complex fibration between symplectic manifolds X, B . If a smooth fiber is uniruled and homologically injective (over \mathbb{R}), then X is uniruled.*

The homologically injective assumption could be a strong one. Notice that for a fiber bundle, the Leray-Hirsch theorem asserts that, under the homologically injective assumption, the homology group of the total space is actually isomorphic to the product of the homology group of the fiber and the base. However, Corollary 3.10 can still be applied for all product bundles, and all projective space fibrations (more generally, if the rational

cohomology ring of a smooth uniruled fiber is generated by the restriction of $[\omega]$.

Moreover, we were informed by McDuff that a Hamiltonian bundle is homologically injective (or equivalently, cohomologically split) if (cf. [37])

a) the base is S^2 (Lalonde-McDuff-Polterovich), and more generally, a complex blow up of a product of projective spaces,

b) the fiber satisfies the hard Lefschetz condition (Blanchard), or its real cohomology is generated by H^2 .

Here is another variation. As in the case of a projective space, for a uniruled manifold up to dimension 4, insertions of a uniruled class can all be assumed to be of the form $[\omega]^i$, thus we also have

Corollary 3.11. *If the general fibers of a possibly singular uniruled fibration are 2-dimensional or 4-dimensional, then the total space is uniruled.*

This in particular applies to a 2-dimensional symplectic conic bundle. A symplectic conic bundle is a conic hypersurface bundle in a smooth \mathbb{P}^k bundle. Holomorphic conic bundles are especially important in the theory of 3-folds. It is conjectured that a projective uniruled 3-fold is either birational to a trivial \mathbb{P}^1 -bundle or a conic bundle.

Another important construction first analyzed by McDuff is the *divisor to ambient space* procedure. It is part of the dichotomy of uniruled divisors and would be discussed in the next section (see Theorem 4.1).

3.3. Geometry. Recall that the symplectic canonical class K_ω of (X, ω) is defined to be $-c_1(TX, J)$ for any ω -tamed almost complex structure J . Observe that for a uniruled manifold K_ω is negative on any uniruled class by a simple dimension computation of the moduli space. In particular, K_ω cannot be represented by an embedded symplectic submanifold. It leads to the following intriguing question.¹

Question 3.12. *Does a uniruled manifold of (real) dimension $2n$ have a negative $K_\omega^i \cdot [\omega]^{n-i}$ for some i ?*

On the other hand, we could ask if the canonical class K_ω is negative in the sense $K_\omega = \lambda[\omega]$ for $\lambda > 0$, is the symplectic manifold uniruled? Such a manifold is known as a monotone manifold in the symplectic category, and it is the analogue of a Fano manifold. Fano manifolds are projectively uniruled by the famous bend-and-break argument of Mori. In fact, Fano manifolds are even rationally connected.

We also observe here that uniruled manifolds satisfy a simple ball packing constraint. To state it let us introduce the notion of a minimal uniruled class, which is a uniruled class with minimal symplectic area among all uniruled classes. This notion will also play a crucial role in subsection 4.1. Then it follows from Gromov's monotonicity argument that the size of any embedded symplectic ball is bounded by the area of a minimal uniruled class.

¹The converse of this question should be compared with the Mumford conjecture: a projective manifold is uniruled if and only if it has Kodaira dimension $-\infty$.

3.4. Rationally connected manifolds. A projective manifold is rationally connected if given any two points p and q there is a rational curve connecting p and q . It is equivalent to “chain rational connected” where there is a chain of rational curves connecting p and q .

The outstanding conjecture is: *A projective manifold X is rationally connected if and only if there is a nonzero connected GW invariant of the form*

$$(5) \quad \langle \tau_{j_1}([pt]), \tau_{j_2}([pt]), \dots, \tau_{j_k}(\beta_k) \rangle_A^X \neq 0$$

for $A \neq 0$.

We define a symplectic manifold to be rationally connected if there is a nonzero invariant of the form (5). Obviously, a rationally connected manifold is uniruled.

Whether symplectic rational connectedness is a birational property appears to be a hard question (cf. [55]), though we believe it is possible to show the invariance under certain types of blow-ups. Characterizing such manifolds is likewise more difficult. But at least we know that all such manifolds in dimension 4 are rational. Moreover, it is expected that symplectic manifold containing a rationally connected symplectic divisor with certain strong positivity is rationally connected (Initial progress has been made in [14]).

Of course we can also similarly define N -rationally connectedness for any integer $N \geq 2$. Moreover, it is not hard to see that there is a parking constraint for N balls in terms of (the area of) a minimal N -rationally connected class.

4. DICHOTOMY OF UNIRULED DIVISORS

We have seen that up to birational cobordism symplectic manifolds are naturally divided into uniruled ones and non-uniruled ones. In this section we discuss uniruled submanifolds of codimension 2, which we simply call symplectic divisors. One motivation comes from the basic fact in algebraic geometry that various birational surgery operations such as contraction and flop have a common feature: the subset being operated on is necessarily uniruled.

Our key observation is that, as in the projective birational program, such a uniruled symplectic divisor admits a dichotomy depending on the positivity of its normal bundle. If the normal bundle is non-negative in certain sense, it will force the ambient manifold to be uniruled. If the normal bundle is negative in certain sense, we can ‘contract’ it to simplify the ambient manifold.

We have a rather general result in the non-negative case. In the second case our progress is limited to simple uniruled divisors in 6-manifolds.

4.1. Dichotomy of uniruled divisor–non-negative half. Suppose that $\iota : D \rightarrow (X, \omega)$ is a symplectic divisor (which we always assume to be smooth). Let N_D be the normal bundle of D in X . Notice that N_D is a

2-dimensional symplectic vector bundle and hence has a well defined first Chern class. We will often use N_D to denote the first Chern class.

Theorem 4.1. *Suppose D is uniruled and A is a minimal uniruled class of D such that*

$$(6) \quad \langle \iota^* \alpha_1, \dots, \iota^* \alpha_l, [pt], \beta_2, \dots, \beta_k \rangle_A^D \neq 0$$

for $\alpha_i \in H^(X; \mathbb{R})$, $\beta_j \in H^*(D; \mathbb{R})$ with $\beta_1 = [pt]$, and $k \leq N_D(A) + 1$. Then (X, ω) is uniruled.*

Here what matters in (6) is the number of insertions which do not come from X . There are situations where can simply take $k = 1$ hence only require $N_D(A)$ be non-negative. In particular, we have

Corollary 4.2. *Suppose D is a homologically injective uniruled divisor of X and the normal bundle N_D is non-negative on a minimal uniruled class. Then X is uniruled.*

We can ask whether the converse of Theorem 4.1 is also true. It is obvious in dimension 4. In higher dimension we could easily construct non-negative singular uniruled divisors. The hard and important question is whether we can they can be smoothed inside X .

As mentioned in the previous section, Theorem 4.1 can also be viewed as a construction of uniruled manifolds, generalizing several early results of McDuff. We list some examples here, more examples can be found in [31].

4-dimensional uniruled divisors: A deep result in dimension 4 is that uniruled manifolds can be completely classified.

Theorem 4.3. ([42], [29], [30], [36], [57]) *A 4-manifold (M, ω) is uniruled if and only if it is rational or ruled. Moreover, the isotopy class of ω is determined by $[\omega]$.*

Here symplectic 4-manifold (M, ω) is called rational if its underlying smooth manifold M is $S^2 \times S^2$ or $\mathbb{P}^2 \# k \overline{\mathbb{P}}^2$ for some non-negative integer k . (M, ω) is called ruled if its underlying smooth manifold M is the connected sum of a number of (possibly zero) $\overline{\mathbb{P}}^2$ with an S^2 -bundle over a Riemann surface.

We need to analyze minimal uniruled classes and the corresponding insertions.

Proposition 4.4. *If A is a uniruled class of a 4-manifold, then A is represented by an embedded symplectic surface, and A satisfies (i) $K_\omega(A) \leq -2$, (ii) $A^2 \geq 0$, (iii) $A \cdot B \geq 0$ for any class B with a non-trivial GW invariant of any genus.*

For \mathbb{P}^2 , let H be the generator of H_2 with positive area. H is a uniruled class and any uniruled class of the form aH with $a > 0$. Obviously, H is the minimal uniruled class. The relevant insertion is $([pt], [pt])$. As $[pt]$ is a restriction class, i.e. an α class, we can take $k = 1$.

Similarly, for the blow-up of an S^2 -bundle over a surface of positive genus, the fiber class is a uniruled class, and any uniruled class is a positive multiple of the fiber class. The relevant insertion for the fiber class is $[pt]$. Thus again we can take $k = 1$.

It is easier to apply Theorem 4.1 in this case.

Corollary 4.5. *Suppose (X^6, ω) contains a divisor D which is diffeomorphic to \mathbb{P}^2 or the blow-up of an S^2 -bundle over a surface of positive genus. If the normal bundle N_D is non-negative on a uniruled class, then X is uniruled.*

For other M^4 , the uniruled classes are not proportional to each other. Thus the minimality condition depends on the class of the symplectic form on M .

We first analyze the easier case of an S^2 -bundles over S^2 . For $S^2 \times S^2$, by uniqueness of symplectic structures, any symplectic form is of product form. Let A_1 and A_2 be the classes of the factors with positive area. It is easy to see that any uniruled class is of the form $a_1 A_1 + b_1 A_2$ with $a_1 \geq 0, a_2 \geq 0$. Thus either A_1 or A_2 has the minimal area.

For the nontrivial bundle $S^2 \tilde{\times} S^2 = \mathbb{P}^2 \# \overline{\mathbb{P}}^2$, let F_0 be the class of a fiber with positive area and E be the unique -1 section class with positive area. If $aF_0 + bE$ is a uniruled class then $b \geq 0$ by (iii) of Proposition 4.4, since $F_0 \cdot E = 1, F_0 \cdot F_0 = 0$. And if $b > 0$, then $a \geq 1$ by (i) of Proposition 4.4. Thus F_0 is always the minimal uniruled class no matter what the symplectic structure is.

Since the relevant insertion for A_1, A_2 and F_0 is just $[pt]$, we have

Corollary 4.6. *Suppose $D = S^2 \times S^2$ and the restriction of the normal bundle N_D to the factor with the least area is non-negative, then X is uniruled. In the case of the non-trivial bundle, X is uniruled if the restriction of the normal bundle N_D to F_0 is non-negative.*

The remaining M^4 are connected sums of \mathbb{P}^2 with at least 2 $\overline{\mathbb{P}}^2$. It is complicated to analyze minimal uniruled classes in this case. In [31] it is shown that they are generated by the so called fiber classes.

Higher dimensional case: In higher dimension, it is still a remote goal to classify all the uniruled symplectic manifolds. Instead of considering an arbitrary uniruled symplectic divisor, we start with Fano hypersurfaces.

When the divisor $D \subset (X, \omega)$ is symplectomorphic to a divisor of \mathbb{P}^n (for $n \geq 4$) of degree at most n , D is Fano and hence uniruled. Of course a particular case is $D = \mathbb{P}^{n-1}$. Since $n \geq 4$, by the Lefschetz hyperplane theorem, $b_2 = 1$ for D . According to Theorem 3.2, for a minimal uniruled class A , we can take k to be equal to 1. Hence X is uniruled if $N_D = \lambda[\omega|_D]$ with $\lambda \geq 0$.

In general case we still need to verify the minimal condition. Of course the uniruled divisor needs not to be a projective manifold. For instances, the divisor could be a rather general uniruled fibration discussed in §2. Let us treat the case of a symplectic \mathbb{P}^k -bundle. Since the line class in the fiber

is uniruled, and the relevant insertions can be taken to be $([pt], [\omega|_D]^k)$, we have

Corollary 4.7. *Suppose D is a symplectic divisor of X . If D is a projective space bundle with the fiber class being the minimal uniruled class and normal bundle N_D being non-negative along the fibers, then X is uniruled.*

McDuff also considered the case of product \mathbb{P}^k -bundles in [44]. A natural source of such a D is from blowing up a ‘non-negative’ \mathbb{P}^k with a large trivial neighborhood. Suppose $\mathbb{P}^k \subset X$ has trivial normal bundle. Then the blow up along \mathbb{P}^k has a divisor $D = \mathbb{P}^k \times \mathbb{P}^{n-k-1}$. The normal bundle of D along a line in \mathbb{P}^k is trivial. Similar to the case of $S^2 \times S^2$, we can argue that the area of this line is minimal among all uniruled class of D . In particular, as observed by [44], a symplectic \mathbb{P}^1 with a sufficiently large product symplectic neighborhood can only exist in a uniruled manifold.

In fact we can prove more.

Corollary 4.8. *Suppose S is a uniruled symplectic submanifold whose minimal uniruled class has area η and insertions all being restriction classes. If S has a trivial symplectic neighborhood of radius at least η . Then X is uniruled.*

4.2. Symplectic ‘blowing-down’ in dimension six. Blowing up in dimension 6 gives rise to a symplectic \mathbb{P}^2 with normal $c_1 = -1$ or a symplectic S^2 -bundle with normal $c_1 = -1$ along the S^2 -fibers. A natural question is whether such a uniruled divisor always arises from a symplectic blow-up. In other words, we are interested in a criterion for blowing-down. A nice feature is that by Theorem 4.3 the answer would also only depend on $[\omega]$. Moreover, as every symplectic structure on such a 4-manifold is Kähler, we can apply algebro-geometric techniques to understand this problem.

The case of \mathbb{P}^2 is simple: it can always be blown down just as in the case of \mathbb{P}^1 with self-intersection -1 in a symplectic 4-manifold. For the case of an S^2 -bundle over a Riemann surface Σ_g of genus g , it is more complicated and perhaps more interesting. Topologically blowing down an S^2 -bundle over Σ_g is the same as topologically fiber summing with the pair of a \mathbb{P}^2 -bundle and an embedded \mathbb{P}^1 -bundle over Σ_g with opposite normal bundle (see 6.1). To perform the fiber sum symplectically we also need to match the symplectic classes of the divisors. For this purpose we need to understand the relative symplectic cone of such a pair. We determine in [32] the relative Kähler cone for various complex structures coming from stable and unstable rank 3 holomorphic bundles over Σ_g . Consequently we obtain

Theorem 4.9. *Let (X, ω) be a symplectic manifold of dimension 6. Let D be a symplectic divisor which is an S^2 -bundle over Σ_g . Suppose $N_D(f) = -1$ and $N_D(s) = d$, where f is the fiber class of D , and s is a section class of D with square 0 if D is a trivial bundle and square -1 if D is a non-trivial bundle. Further assume that $[\omega|_D](f) = a$ and $[\omega|_D](s) = b$. Then a*

symplectic fiber sum can be performed with a symplectic $(\mathbb{P}^2, \mathbb{P}^1)/\Sigma_g$ pair if either $d \geq 0$, or

- (i) $d < 0, g = 0, b > [\frac{-d+2}{3}]a$,
- (ii) $d < 0, g \geq 1, b > \frac{-d}{3}a$.

Here $[x]$ denotes the largest integer bounded by x from above. This result is optimal in the case of genus 0. For instance, we show that a symplectic $S^2 \times S^2$ with normal $c_1 = -1$ along each family of S^2 can be blown down if the symplectic areas of the two factors are not the same. This picture is consistent with the flop operation for projective 3-folds. On the other hand, when $g \geq 1$, a symplectic $S^2 \times \Sigma_g$ with normal $c_1 = -1$ along each factor can be fiber summed if the symplectic area of the S^2 factor is at most 3 times of that of the Σ factor. We would very much like to find out whether the restriction on the areas is really necessary in the case of positive genus. The picture would be rather nice if the restriction can eventually be removed. On the other hand, it would be surprising, even intriguing, if it turns out there is an obstruction. We are also interested to see how much of the 6-dimensional investigation can be carried out to higher dimensions.

Another remaining issue is whether such a symplectic fiber sum is actually a birational cobordism operation. This is because that a symplectic blowing down further requires the symplectic $(\mathbb{P}^2, \mathbb{P}^1)/\Sigma_g$ pair have an infinity symplectic section, which is a symplectic surface of genus g . We are investigating whether our more general fiber sums are equivalent to symplectic blowing down up to deformation. Either positive or negative answer would be very interesting.

5. MINIMAL SYMPLECTIC MANIFOLDS

5.1. Minimality. Motivated by the Mori program for algebraic 3-folds and understandings of symplectic 4-manifolds (and 2-manifolds), we discuss in this section the notion of minimal manifolds in dimension 6.

Let us first recall the notion of minimality by McDuff in dimension 4. Let \mathcal{E}_X be the set of homology classes which have square -1 and are represented by smoothly embedded spheres. We say that X is smoothly minimal if \mathcal{E}_X is empty. Let $\mathcal{E}_{X,\omega}$ be the subset of \mathcal{E}_X which are represented by embedded ω -symplectic spheres. We say that (X, ω) is symplectically minimal if $\mathcal{E}_{X,\omega}$ is empty. When (X, ω) is non-minimal, one can blow down some of the symplectic -1 spheres to obtain a minimal symplectic 4-manifold (N, μ) , which is called a (symplectic) minimal model of (X, ω) ([Mc]). We summarize the basic facts about the minimal models in the following proposition.

Theorem 5.1 ([28], [30], [42], [57]). *Let X be a closed oriented smooth 4-manifold and ω a symplectic form on X compatible with the orientation of X .*

1. *X is smoothly minimal if and only if (X, ω) is symplectically minimal. In particular the underlying smooth manifold of the (symplectic) minimal model of (X, ω) is smoothly minimal.*

2. If (X, ω) is not rational nor ruled, then it has a unique (symplectic) minimal model. Furthermore, for any other symplectic form ω' on X compatible with the orientation of X , the (symplectic) minimal models of (X, ω) and (X, ω') are diffeomorphic as oriented manifolds.
3. If (X, ω) is rational or ruled, then its (symplectic) minimal models are diffeomorphic to CP^2 or an S^2 -bundle over a Riemann surface.

Definition 5.2. A symplectic 6-manifold is minimal if it does not contain any rigid stable uniruled divisor.

Here, a uniruled divisor is stable if one of its uniruled classes A is a GW class of the ambient manifold with $K_\omega(A) \leq -1$. And a uniruled divisor is rigid if none of its uniruled class is a uniruled class of the ambient manifold.

We observe this definition also applies to manifolds of dimensions 2 and 4. Every 2-manifold is obviously minimal as the only divisors are points. For a 4 manifold any uniruled divisor is S^2 . Let A be the class of a stable S^2 . Then $K_\omega(A) \leq -1$. On the other hand a rigid S^2 must have $K_\omega(A) < 0$ by [42]. The only rigid stable uniruled divisor is an S^2 with $K_\omega(A) = -1$. By the adjunction formula it is a symplectic -1 sphere. Thus this definition of minimality agrees with the usual notion of minimality by McDuff.

An immediate question is whether any 6-manifold has a minimal model. A general strategy is to construct a minimal model by performing consecutive contractions as discussed in 4.2. Notice that we do not need to eliminate all negative uniruled submanifolds, only those which are rigid and stable. Thus we might be able to avoid complicated singularities. However, we will definitely encounter orbifold singularities. The first author showed in [50] that all $(K$ -negative) extremal rays for algebraic 3-folds give rise to non-trivial GW classes. One such an extremal ray arises from the line class of a \mathbb{P}^2 divisor with normal $c_1 = -2$. To carry out the contraction one has to enlarge to the category of symplectic orbifolds.

Now we single out an important class of minimal manifolds.

Definition 5.3. We define a cohomology class $\alpha \in H^2(X, \mathbb{Z})$ to be nef if it is non-negative on all GW classes.

Lemma 5.4. Let (X, ω) be a manifold with nef K_ω . Then (M, ω) is non-uniruled and minimal.

Proof. The first statement is obvious as any uniruled class A of X satisfies $K_\omega(A) \leq -2$.

There can not be any stable uniruled divisor in X as $K_\omega(A) \leq -1$ for class A which is a uniruled class of a stable divisor as well as a GW class of X .

□

A minimal model of a uniruled manifold is still uniruled and hence can not be K_ω -nef. The natural question is whether any minimal model of a non-uniruled manifold must have nef K_ω . The first step towards this question

is to show that any GW class A with $K_\omega(A) \leq -1$ is a uniruled class of a (smooth) divisor. Here the issue is again smoothing singular uniruled divisors.

A further question is whether birational cobordant K_ω -nef manifolds are related by an analogue of the K -equivalence. Two algebraic manifolds X, X' are K -equivalent if there is a common resolutions $\pi_1 : Z \rightarrow X, \pi_2 : Z \rightarrow X'$ such that $\pi_1^* K_X = \pi_2^* K_{X'}$. K -equivalent manifolds have many beautiful properties, in particular, they have the same betti numbers.

5.2. Kodaira dimension. The notion of Kodaira dimension has been defined for symplectic manifolds up to dimension 4 ([26], [47]). Whenever it is defined it is a finer invariant of birational cobordism than uniruledness.

The Kodaira dimension of a 2-dimensional symplectic manifold (F, ω) is defined as

$$\kappa(F, \omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0. \end{cases}$$

Clearly $\kappa(F, \omega) = -\infty, 0, 1$ if and only if the genus of F is $0, 1, \geq 2$ respectively. Notice that $\kappa(F, \omega) = -\infty$ if and only if (F, ω) is uniruled.

For a minimal symplectic 4-manifold (X, ω) its Kodaira dimension is defined in the following way ([21], [47], [26]):

$$\kappa(X, \omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0. \end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Based on the Seiberg-Witten theory and properties of minimal models (cf. Theorem 5.1, [30], [24], [48]), [57]) it is shown in [26] that the Kodaira dimension $\kappa(M, \omega)$ is well defined. In particular, we need to check that a minimal 4-manifold cannot have

$$(7) \quad K_\omega \cdot [\omega] = 0, K_\omega \cdot K_\omega > 0.$$

We list some basic properties of $\kappa(X, \omega)$. It is also observed in [26] that, if ω is a Kähler form on a complex surface (X, J) , then $\kappa(X, \omega)$ agrees with the usual holomorphic Kodaira dimension of (X, J) .

(X, ω) has $\kappa = -\infty$ if and only if it is uniruled.

It is further shown in [26] that minimal symplectic 4-manifolds with $\kappa = 0$ are exactly those with torsion canonical class, thus they can be viewed as *symplectic Calabi-Yau surfaces*. Known examples of symplectic 4-manifolds with torsion canonical class are either Kähler surfaces with (holomorphic) Kodaira dimension zero or T^2 -bundles over T^2 . It is shown in [27] and [2] that a *minimal* symplectic 4-manifold with $\kappa = 0$ has the rational homology as that of K3 surface, Enriques surface or a T^2 -bundle over T^2 .

Suppose (X, ω) is a minimal 6-dimensional manifold. we propose to define its Kodaira dimension in the following way²:

$$\kappa(X, \omega) = \begin{cases} -\infty & \text{if one of } K_\omega^i \cdot [\omega]^{3-i} \text{ is negative,} \\ k & \text{if } K_\omega^i \cdot [\omega]^{3-i} = 0 \text{ for } i \geq k, \text{ and } K_\omega^i \cdot [\omega]^{3-i} > 0 \text{ for } i \leq k. \end{cases}$$

Notice that, as in dimension 4, there is the issue of well definedness of $\kappa(X, \omega)$. And this leads to some possible intriguing properties of minimal 6-manifolds, one of which is whether there is any minimal 6-manifold with

$$(8) \quad K_\omega \cdot [\omega]^2 = 0, K_\omega^2[\omega] = 0, K_\omega^3 > 0.$$

6. CORRESPONDENCES IN THE GROMOV-WITTEN THEORY

As the Gromov-Witten theory is built into the foundation of symplectic birational geometry, it is natural that we use many techniques from the Gromov-Witten theory such as localization and degenerations. It turns out that we have to use very sophisticated Gromov-Witten machinery. Take the birational invariance of the uniruledness as an example. The definition of uniruledness requires only a single non-vanishing GW invariant. However, it is well-known that a single Gromov-Witten invariant tends to transform in a rather complicated fashion. On the other hand, it is often easier to control the transformation of the Gromov-Witten theory as a whole. One often phrases such a amazing phenomenon as a kind of correspondences. There are many examples such as the Donaldson-Thomas/Gromov-Witten correspondence [40], the crepant resolution conjecture [52] and so on.

The correspondences appearing in our context are not as strong as the above ones. Its first example is the “relative/absolute correspondence” constructed by Maulik-Okounkov-Pandharipande ([41]). It is the generalization to the situation of blow-up/down by the authors and Hu which underlies the birational invariance of uniruledness. And Theorem 4.1 is proved by another technical variation of the relative/absolute correspondences incorporating divisor invariants. Roughly speaking, a correspondence in this context is a package to organize the degeneration formula in a very nice way.

We restrict ourselves to genus GW invariants in this article.

6.1. Symplectic cut and the degeneration formula.

6.1.1. *Symplectic cut along a submanifold.* Let (X, ω) be a closed symplectic manifold. Let S be a hypersurface having a neighborhood with a free Hamiltonian S^1 -action. For instance, if there is a symplectic submanifold in X , then the hypersurfaces corresponding to sphere bundles of the normal bundle have this property. Let Z be the symplectic reduction at the level S , then Z is the S^1 -quotient of S and is a symplectic manifold of 2 dimension less.

²Compare with Question 3.12

We can cut X along S to obtain two closed symplectic manifolds $(\overline{X}^+, \omega^+)$ and $(\overline{X}^-, \omega^-)$ each containing a smooth copy of Z , and satisfying $\omega^+|_Z = \omega^-|_Z$ ([20]).

In particular, the pair (ω^+, ω^-) defines a cohomology class of $\overline{X}^+ \cup_Z \overline{X}^-$, denoted by $[\omega^+ \cup_Z \omega^-]$. Let p be the continuous collapsing map

$$p : X \rightarrow \overline{X}^+ \cup_Z \overline{X}^-.$$

It is easy to observe that

$$(9) \quad p^*([\omega^+ \cup_Z \omega^-]) = [\omega].$$

Let $\iota : D \rightarrow X$ be a smooth connected symplectic divisor. Then we can cut along D , or precisely, cut along a small circle bundle S over D inside X .

In this case, as a smooth manifold, $\overline{X}^+ = X$, which we will denote by \tilde{X} . Denote the symplectic reduction of S in \tilde{X} still by D . Notice however, the symplectic structure is different from the original divisor. And $\overline{X}^- = \mathbb{P}(N_D \oplus \mathbb{C})$, the projectivization of $\mathbb{P}(N_D \oplus \mathbb{C})^3$. We will often denote it simply by P_D or P . Notice that $\mathbb{P}(N_D \oplus \mathbb{C})$ has two natural sections,

$$D_0 = \mathbb{P}(0 \oplus \mathbb{C}), \quad D_\infty = \mathbb{P}(N_D \oplus 0).$$

The symplectic reduction of S in P_D is the section D_∞ .

In summary, in this case, X degenerates into (\tilde{X}, D) and (P_D, D_∞) . We also denote ω^- by ω_P .

More generally, we can cut along a symplectic submanifold Q of codimension $2k$, or precisely, cut along a sphere bundle S over Q . Then \overline{X}^+ is a symplectic blow up of X along Q and the symplectic reduction $Z \subset \overline{X}^+$ is the exceptional symplectic divisor. In this case $\overline{X}^- = \mathbb{P}(N_Q \oplus \mathbb{C})$, which is a \mathbb{P}^k bundle over Y .

6.1.2. Degeneration formula. Given a symplectic cut, there is a basic link between absolute invariants of X and relative invariants of (\overline{X}^\pm, Z) in [33] (see also [15], and [23] in algebraic geometry). We now describe such a formula.

Let $B \in H_2(X; \mathbb{Z})$ be in the kernel of

$$p_* : H_2(X; \mathbb{Z}) \longrightarrow H_2(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{Z}).$$

By (9) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. For $A \in H_2(X; \mathbb{Z})$ define $[A] = A + \text{Ker}(p_*)$ and

$$(10) \quad \langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{[A]}^X = \sum_{B \in [A]} \langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_B^X.$$

At this stage we need to assume that each cohomology class α_i is of the form

$$(11) \quad \alpha_i = p^*(\alpha_i^+ \cup_Z \alpha_i^-).$$

³Notice that our convention here is opposite to that in [13]

Here $\alpha_i^\pm \in H^*(\overline{X}^\pm; \mathbb{R})$ are classes with $\alpha_i^+|_Z = \alpha_i^-|_Z$ so that they give rise to a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{R})$.

The degeneration formula expresses $\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{[A]}^X$ as a sum of products of relative invariants of (\overline{X}^+, Z) and (\overline{X}^-, Z) , possibly with disconnected domains. In each product of relative invariants, what is relevant for us are the following conditions:

- the union of two domains along relative marked points is a stable genus 0 curve with k marked points,
- the total curve class is equal to $p_*(A)$,
- the relative insertions are dual to each other,
- if α_i^+ appears for i in a subset of $\{1, \dots, k\}$, then α_j^- appears for j in the complementary subset of $\{1, \dots, k\}$.

In the case of cutting along a symplectic submanifold it is easy to show that all the invariants on the right hand side of (10) vanish except

$$\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_A^X.$$

Thus the degeneration formula computes $\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_A^X$ in terms of relative invariants of (\overline{X}^\pm, Z) .

6.2. Absolute/Relative, blow-up/down and divisor/ambient space correspondences. Giving a symplectic manifold (X, ω) we are interested in determining its GW classes and uniruled classes. Suppose X has some explicit symplectic submanifolds, then we could cut X and attempt to apply the degeneration formula to compute a given GW invariant. However, this is often impractical, as we need to know all the relevant relative invariants of (\overline{X}^\pm, Z) and relative invariants are generally harder to compute themselves.

Remarkably it is shown in [41] that, in case the submanifold D is a divisor and hence $\overline{X}^+ = X$, the degeneration formula can be inverted to express a (non-descendant) relative invariant of (X, D) in terms of invariants of X and relative invariants of (P_D, D_∞) . We briefly describe the strategy of proof in [41].

The first idea is to associate a possibly descendant invariant of X to each non-descendant relative invariant of (X, D) where absolute insertions are kept intact and contact orders of relative insertions are replaced by appropriate descendant powers.

Observe then relative GW invariants are linear on the insertions. So we can choose a generating set I of non-descendant relative GW invariants by choose bases of cohomology of X and D and require the absolute and relative insertions lie in the two bases.

The next idea is to introduce a partial order on I with 2 properties. Firstly, given a relative invariant of (X, D) , when applying the degeneration formula to the associated invariant of X , the given relative invariant is the largest one among those relative invariants of (X, D) appearing in the formula and with nonzero coefficient. Recall that the right hand side of the

degeneration formula is a sum of products, for each product the relative invariant of (P_D, D_∞) is considered to be the coefficient. Secondly, the partial order is lower bounded in the sense there are only finitely many invariants in I lower than any given relative invariant in I .

Then inductively, any relative invariant in I can be expressed in terms of invariants of X and relative invariants of (P_D, D_∞) .

In [13] we slightly reformulate the absolute/relative correspondence as a lower bounded and triangular (and hence invertible) transformation T in an infinity dimension vector space, sending the relative vector v_{rel}^I determined by all relative invariants in I to the v_{abs}^I absolute vector determined by all the associated invariants of X . Furthermore, if I_{pt} is the subset of I such that one of the absolute insertions is a $[pt]$ insertion, T still interchanges the absolute and relative subvectors v_{abs}^{pt} and v_{rel}^{pt} determined by I_{pt} .

We further generalize the absolute/relative correspondence to the more general cuts along submanifolds of arbitrary codimension to obtain the blow-up/down correspondence. Let \tilde{X} be a blow-up of X along a submanifold Q with exceptional divisor D . We can cut \tilde{X} along D as well as cut X along Q . It is important to observe that the $+$ pairs of these 2 cuts are essentially the same as the pair (\tilde{X}, D) , in particular, they have the same relative invariants. Another important fact is that each invariant of \tilde{X} in $v_{abs}^{I_{pt}}(\tilde{X})$ has a $[pt]$ insertion, and the same is true for X . In fact, the converse is also true. Thus \tilde{X} is uniruled if and only if the absolute vector $v_{abs}^{I_{pt}}(\tilde{X})$ is nonzero, and the same for X .

We now explain why the birational invariance of uniruledness is an immediate consequence. Suppose the blow-up \tilde{X} is uniruled, then $v_{abs}^{I_{pt}}(\tilde{X})$ is nonzero. Hence the relative vector $v_{rel}^{I_{pt}}(\tilde{X})$ is nonzero by the absolute/relative correspondence. Apply now the blow-up/down correspondence to conclude that $v_{abs}^{I_{pt}}(X)$ is nonzero. Therefore X is uniruled as well. Similarly we can obtain the reverse direction.

In the case that the submanifold D is a divisor, another variation of the absolute/relative correspondence, the so called sup-admissible correspondence is established in [31]. For this correspondence the relative vector is enlarged to include relative invariants of (P_D, D_∞) with curve classes in the image of $\iota_* : H_2(D) \rightarrow H_2(M)$.

In the case the submanifold D is a uniruled divisor satisfying the condition of Theorem 4.1, it is further shown in [31] that the sup-admissible correspondence can be restricted to the subvector with a relative $[pt]$ insertion and with the curve class constrained to have symplectic area bounded above by that of a minimal uniruled class of D . The proof is rather complicated. It involves the reduction scheme of relative invariants of \mathbb{P}^1 -bundle to invariants of the base in [41] as well as [3] to prove certain vanishing results of relative invariants of (P_D, D_∞) . To prove Theorem 4.1, we also need a non-vanishing result of relative invariants of (P_D, D_∞) to get the divisor/ambient space correspondence. The outcome of this correspondence is a

nonzero vector of invariants of X , each invariant containing a $[pt]$ insertion. Hence X is uniruled.

7. CONCLUDING REMARKS

Readers can clear sense that the subject of symplectic birational geometry is only at its beginning. Constructing uniruled manifolds by symplectic methods remains to be a challenging problem. We have already mentioned the issue of singularities: a divisorial contraction in dimension six already introduces orbifold singularities. We expect that our program can be carried over to orbifolds.

New areas of research include the dichotomy of higher codimension uniruled submanifolds and transitions. We ponder whether it makes sense to view transitions as extended birational equivalences. Understanding these questions requires new ideas and technologies.

7.1. Dichotomy of uniruled submanifold of higher codimension. We have a good knowledge of the dichotomy of uniruled symplectic divisors at least in dimension six. It is natural that we want to expand our understanding to higher codimension uniruled submanifolds. There are many reasons to believe that the higher codimension case is very different from the divisor case. In algebraic geometry, this is where we encounter other more subtle surgeries such as flip and flop. In the symplectic category, this is where Gromov's h-principle is very effective. Therefore, higher codimensional uniruled submanifolds should provide a fertile ground for these two completely different theories to interact.

Corollary 4.8 strongly indicates that the *size* of a maximal neighborhood should play an important role. Such a phenomenon was first observed in McDuff [45]. One could also wonder whether convexity also plays a role (compare with [19]). It is also desirable to define stable and rigid uniruled submanifolds. Right now, this is largely an unknown and exciting territory.

7.2. Transition. Recall that a transition in the holomorphic category interchanges a resolution with a smoothing. Symplectically, a smoothing can be thought of as gluing with a neighborhood of a configuration of Lagrangian spheres (vanishing cycles). In particular, a simplest symplectic transition interchanges a symplectic submanifold with a Lagrangian sphere. However, a transition is in general not a birational operation. Thus an important question is to construct symplectic transitions which are birational cobordism operations. Such transitions will enhance our ability to 'contract' stable uniruled divisors (submanifolds).

For general transitions it was conjectured by the second author in [51] that the quantum cohomology behaves nicely. We could similarly ask whether uniruledness is preserved under general transitions. Correspondences have

been very successful to keep track of the total transformation of Gromov-Witten theory under birational equivalences. A natural problem is to construct GW correspondence for transitions. This is a new territory for the Gromov-Witten theory.

The most famous example is the conifold transition. Geometrically, we replace a holomorphic 2-sphere with a Lagrangian 3-sphere. The conifold transition plays an important role in the theory of Calabi-Yau 3-folds and string theories. In this case one could partially verify the invariance of uniruledness by [33].

To build a full correspondence we may have to enlarge the usual Gromov-Witten theory to the so called open Gromov-Witten theory to allow holomorphic curves with boundary in Lagrangian manifolds. This is the well-known *open-closed* duality in physics. It also raises an possibility to extend symplectic birational geometry to the *open* birational geometry. For example, we can define a symplectic manifold to be *open uniruled* if it contains a nonzero genus zero possibly open GW-invariant with a $[pt]$ insertion. Such a notion has already been studied in symplectic geometry (cf. [7]). Further investigation will greatly expand our horizon to understand symplectic birational geometry.

7.3. Final question. We feel that symplectic birational geometry is an interesting subject and have raised many questions. We finish this survey with one more: *what kind of structures of symplectic manifold are detected by higher genus GW invariants?*

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